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Spectral Density of a Nonlinear Function of a Gaussian Process

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We derive an expression which can be used in a relatively straightforward manner to obtain either the autocorrelation function or the spectral density of any "reasonable" function of any stationary gaussian process. The expression is used to study the spectral density of a sinusoidal wave which is phase modulated by a hard-limited gaussian process.

I. INTRODUCTION

A band-limited gaussian process having a rectangular power spectrum is assumed for some purposes to be an adequate approximation for several classes of modulating signals encountered in phase modulation (PM) systems.[1] However, in an actual implementation of a PM system, the modulating signal passes through circuitry which saturates when the signal rises above a fixed level. This clipping (i.e., limiting) level is usually adjusted fairly high (nominally at four times the rms of the modulating signal) and then ignored in any subsequent analysis of the system. Consequently, the objective of this study is to determine the qualitative effect of hard-limiting the modulating signal in a PM system.

From a mathematical viewpoint, we can obtain an understanding of the preceding question by investigating the following problem: Find the spectral density of a sinusoidal wave which is phase modulated by a function $g(X_t)$ of the stationary gaussian process X_t. Of course, this version of the problem can also be viewed as finding the spectral density of a (composite) nonlinear function of a gaussian process; a problem originally studied by S. O. Rice,[2] D. Middleton[3] and W. R. Bennett.[4] In fact, we do use their approach (representing the nonlinearity in terms of a transform) to derive an expression which is essentially the starting point of our analysis. However, using our relation avoids some of the complexity associated with the transform method. A derivation of the expression is given in Appendix A.

The notation and general results are presented in Section II along with two examples. In the third section, we obtain specific results for the hard-limiting case.

II. SPECTRAL DENSITY OF A PM WAVE MODULATED BY A NONLINEAR FUNCTION OF A GAUSSIAN PROCESS

Let $W(t)$ denote a constant-amplitude sinusoidal wave which is phase modulated by a real-valued function $g(X_t)$ of the stationary gaussian process X_t. That is,

$$W(t) = A \cos(\omega_c t + g(X_t) + \theta) \tag{1}$$

where A is the wave amplitude, $f_c = \omega_c/2\pi$ is the carrier frequency, and θ is a random variable with probability density function

$$\pi_\theta(\theta) = \begin{cases} \dfrac{1}{2\pi} & \text{for} \quad 0 \leqq \theta < 2\pi, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

To obtain the spectral density for $W(t)$, it is convenient to express the wave in terms of complex variables as[5]

$$W(t) = \mathrm{Re}\,\{\tilde{W}(t)\} \tag{3}$$

where†

$$\tilde{W}(t) = A \exp[j(\omega_c t + \theta)]V(t), \tag{4}$$

and

$$V(t) = \exp[jg(X_t)]. \tag{5}$$

Since X_t is stationary and θ is uniformly distributed, both $\tilde{W}(t)$ and $V(t)$ are wide-sense stationary.[5] Moreover, the spectral density $S_w(f)$ of $W(t)$ is given by

$$S_w(f) = \tfrac{1}{4}[S_v(f - f_c) + S_v(-f - f_c)] \tag{6}$$

where $S_v(f)$ is the spectral density of $V(t)$.[5]

If we define $R_v(\tau) = \langle V(t + \tau)\overline{V(t)}\rangle$, the Wiener–Khintchine theorem implies that

$$S_v(f) = \int_{-\infty}^{\infty} R_v(\tau) \exp[-j2\pi f\tau]\, d\tau. \tag{7}$$

† We use Re (z) and Im (z) to denote respectively the real and imaginary parts of the complex variable z. The symbol $\bar{z}$ denotes the complex conjugate of z.

So, assume that $X_{t+\tau}$ and X_t are gaussian with joint probability density function

$$p(x_1, x_2) = \frac{1}{2\pi\Gamma^2\sqrt{1-r^2}} \exp\left(-\frac{x_1^2 - 2rx_1x_2 + x_2^2}{2\Gamma^2(1-r^2)}\right),$$

$$-\infty < x_i < \infty, \qquad (8)$$

where $R_x(\tau) = \langle X_{t+\tau}X_t\rangle$, $\Gamma^2 = R_x(0)$ and $r = R_x(\tau)/\Gamma^2$. It follows that

$$R_v(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\,[j(g(x_1) - g(x_2))]p(x_1, x_2)\,dx_1\,dx_2\,. \qquad (9)$$

In Appendix A, we show that if $G(x)$ is of exponential order and of bounded variation on bounded intervals,† then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x_1)\overline{G(x_2)}p(x_1, x_2)\,dx_1\,dx_2 = \sum_{n=0}^{\infty} a_n r^n, \qquad |r| \leqq 1, \qquad (10)$$

where

$$a_n = \frac{\Gamma^{2n}}{n!}\left|\int_{-\infty}^{\infty} G(x)\left(\frac{d^n}{dx^n}p(x)\right)dx\right|^2,$$

and

$$p(x) = \frac{1}{\sqrt{2\pi\Gamma^2}}\exp\left(-\frac{x^2}{2\Gamma^2}\right)$$

is the probability density for X_t. Setting $G(x) = \exp[jg(x)]$,

$$R_v(\tau) = \sum_{n=0}^{\infty} C_n r^n, \qquad (11)$$

where‡

$$C_n = \frac{\Gamma^{2n}}{n!}\left|\int_{-\infty}^{\infty} \exp\,[jg(x)]\left[\frac{d^n}{dx^n}p(x)\right]dx\right|^2. \qquad (12)$$

Recall that the Hermite polynomial of degree n, $H_n(x)$, satisfies the relation

$$\frac{d^n}{dx^n}\exp(-x^2) = (-1)^n H_n(x)\exp(-x^2).$$

† We define such functions to be "reasonable."

‡ Since $|\exp\,[jg(x)]| = 1$ for all finite $g(x)$, it follows that equations (11) and (12) are valid for all functions $g(x)$ which are of bounded variation on finite intervals. This should include most functions which one might encounter in engineering applications.

Consequently,

$$C_n = \frac{1}{2^n n!} \left| \int_{-\infty}^{\infty} \exp\,[jg(x)] H_n\left(\frac{x}{\sqrt{2}\,\Gamma}\right) p(x)\, dx \right|^2 . \tag{13}$$

Since the coefficients $\{C_n\}$ are independent of $R_x(\tau)$, an infinite series representation for the spectral density $S_v(f)$ can be obtained from equations (7) and (11). To obtain the series, let

$$S_1(f) = \int_{-\infty}^{\infty} R_x(\tau) \exp\,(-j2\pi f\tau)\, d\tau \tag{14}$$

and define $S_n(f)$ to be the n-fold convolution of $S_1(f)$ with itself; i.e., $S_n(f) = S_{n-1}(f) * S_1(f)$ for $n \geqq 2$. It follows that

$$S_v(f) = C_0\, \delta(f) + \sum_{n=1}^{\infty} C_n \frac{S_n(f)}{\Gamma^{2n}}, \qquad -\infty < f < \infty . \tag{15}$$

Using equations (13) and (15), one can see that the carrier power is given by

$$C_0 = |\langle \exp\,[jg(X_t)] \rangle|^2 . \tag{16}$$

2.1 *Examples*

To obtain a better understanding of the preceding results, we present two examples. The first example uses $g(x) = x$ (and serves as a partial check on our results since this case is well known[5]). The second example assumes extreme clipping,

$$g(x) = \begin{cases} b & \text{for} \quad x > 0, \\ -b & \text{for} \quad x < 0, \end{cases}$$

and serves as a preview for the hard-limiting case presented in Section III.

2.1.1 *Phase Modulation*

For this example, $g(x) = x$ for all x. Consequently, from equation (13) we have, for $n \geqq 0$,

$$C_n = \frac{1}{2^n n!} \left| \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\,(j\sqrt{2}\,\Gamma x) H_n(x) \exp\,(-x^2)\, dx \right|^2 . \tag{17}$$

Thus,

$$C_{2n} = \frac{1}{2^{2n}(2n)!} \left| \frac{2}{\sqrt{\pi}} \int_0^{\infty} \cos\,(\sqrt{2}\,\Gamma x) H_{2n}(x) \exp\,(-x^2)\, dx \right|^2 , \tag{18}$$

$$C_{2n} = \exp(-\Gamma^2) \frac{(\Gamma^2)^{2n}}{(2n)!}, \tag{19}$$

(see Ref. 6, Section 7.388-3).

In a similar fashion $C_{2n+1} = \exp(-\Gamma^2)(\Gamma^2)^{2n+1}/(2n+1)!$, which implies that

$$C_n = \exp(-\Gamma^2) \frac{\Gamma^{2n}}{n!} \quad \text{for} \quad n \geqq 0. \tag{20}$$

Now, substitute equation (20) into equation (11) to obtain the well-known result[5]

$$R_s(\tau) = \exp[-\Gamma^2(1 - r)]. \tag{21}$$

2.1.2 *Phase Modulation by an Extreme-Limited Gaussian Process*

For this case,

$$g(x) = \begin{cases} b & \text{if} \quad x > 0, \\ -b & \text{if} \quad x < 0. \end{cases} \tag{22}$$

Using equation (22) in equation (13), one obtains

$$C_n = \frac{1}{2^n n!} \left| \exp(jb) \int_0^\infty H_n\left(\frac{x}{\sqrt{2}\,\Gamma}\right) p(x)\, dx \right.$$
$$\left. + \exp(-jb) \int_{-\infty}^0 H_n\left(\frac{x}{\sqrt{2}\,\Gamma}\right) p(x)\, dx \right|^2, \tag{23}$$

which reduces to

$$C_{2n} = \frac{(\cos(b))^2}{2^{2n}(2n)!} \left(\frac{2}{\sqrt{\pi}} \int_0^\infty H_{2n}(x) \exp(-x^2)\, dx \right)^2 \tag{24}$$

and

$$C_{2n+1} = \frac{(\sin(b))^2}{2^{2n+1}(2n+1)!} \left(\frac{2}{\sqrt{\pi}} \int_0^\infty H_{2n+1}(x) \exp(-x^2)\, dx \right)^2, \tag{25}$$

for $n \geqq 0$. Since

$$\int_0^\infty \exp(-y^2) H_n(y)\, dy = H_{n-1}(0) \quad \text{for} \quad n \geqq 1, \tag{26}$$

we have

$$C_0 = (\cos(b))^2, \tag{27}$$

$$C_{2n} = 0 \quad \text{for} \quad n \geqq 1, \tag{28}$$

and

$$C_{2n+1} = \frac{(\sin (b))^2}{2^{2n+1}(2n+1)!}\left(\frac{2}{\sqrt{\pi}} H_{2n}(0)\right)^2. \tag{29}$$

Using the fact that $H_0(0) = 1$ and $H_{2n}(0) = (-1)^n 2^n [1 \cdot 3 \cdots (2n-1)]$ for $n \geqq 1$, we see that

$$C_1 = \frac{2}{\pi} (\sin (b))^2$$

and

$$C_{2n+1} = \frac{2}{\pi} (\sin (b))^2 \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n n!\,(2n+1)} \quad \text{for} \quad n \geqq 1. \tag{30}$$

Substituting equations (27), (28) and (30) into equation (11) and recalling the series expansion for Arcsin x, one obtains

$$R_s(\tau) = (\cos (b))^2 + (\sin (b))^2 \frac{2}{\pi} \operatorname{Arcsin}\left(\frac{R(\tau)}{R(0)}\right), \tag{31}$$

for $-\infty < \tau < \infty$.

Notice that the carrier power, $C_0 = (\cos (b))^2$, vanishes if $b = \pi/2 + k\pi$, $k = 0, 1, \cdots$, while all the power goes into the carrier whenever $b = k\pi$, $k = 0, 1, \cdots$. If $b \neq k\pi$, the continuous part of the spectrum is simply a scaled version of the spectrum for an amplitude-modulated wave when the modulator is an extreme-limited gaussian process. This problem was studied by J. H. VanVleck.[7]

III. PHASE MODULATION BY A HARD-LIMITED GAUSSIAN PROCESS

A problem of interest arises when the function g represents an ideal hard-limiter; i.e.,

$$g(x) = \begin{cases} b & \text{for} \quad x > b, \\ x & \text{for} \quad |x| \leqq b, \\ -b & \text{for} \quad x < -b, \end{cases} \tag{32}$$

for some $b \geqq 0$.

3.1 *Carrier Power*

Noting equations (13) and (16), one can see that the carrier power is given by

$$C_0 = \left(\frac{1}{\sqrt{2\pi\Gamma^2}} \int_{-b}^{b} \cos(x) \exp\left(-\frac{x^2}{2\Gamma^2}\right) dx + \cos(b) \frac{2}{\sqrt{2\pi\Gamma^2}} \int_{b}^{\infty} \exp\left(-\frac{x^2}{2\Gamma^2}\right) dx\right)^2.$$

Defining $\alpha = b/\Gamma$ (the relative clipping level), we have

$$C_0 = \left(\frac{2}{\sqrt{\pi}} \int_0^{\alpha/\sqrt{2}} \cos(\sqrt{2}\,\Gamma x) \exp(-x^2)\, dx + \cos(b) \operatorname{erfc}\left(\frac{\alpha}{\sqrt{2}}\right)\right)^2 \tag{33}$$

where

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^{z} \exp(-t^2)\, dt \tag{34}$$

is the error function[8] and $\operatorname{erfc}(z) = 1 - \operatorname{erf}(z)$.

A more useful expression for C_0 can be obtained in terms of the error function with complex argument. Apparently,

$$\frac{2}{\sqrt{\pi}} \int_0^{\alpha/\sqrt{2}} \cos(\sqrt{2}\,\Gamma x) \exp(-x^2)\, dx$$

$$= \operatorname{Re}\left\{\frac{1}{\sqrt{\pi}} \int_{-\alpha/\sqrt{2}}^{\alpha/\sqrt{2}} \exp(-j\sqrt{2}\,\Gamma x - x^2)\, dx\right\}, \tag{35}$$

$$= \exp\left(-\frac{\Gamma^2}{2}\right) \operatorname{Re}\left\{\frac{1}{\sqrt{\pi}} \int_{-\alpha/\sqrt{2}}^{\alpha/\sqrt{2}} \exp\left(-\left(x + j\frac{\Gamma}{\sqrt{2}}\right)^2\right) dx\right\}, \tag{36}$$

$$= \frac{1}{2} \exp\left(-\frac{\Gamma^2}{2}\right) \operatorname{Re}\left\{\operatorname{erf}\left(\frac{\alpha}{\sqrt{2}} + j\frac{\Gamma}{\sqrt{2}}\right) - \operatorname{erf}\left(\frac{\alpha}{\sqrt{2}} + j\frac{\Gamma}{\sqrt{2}}\right)\right\}, \tag{37}$$

$$= \exp\left(-\frac{\Gamma^2}{2}\right) \operatorname{Re}\left\{\operatorname{erf}\left(\frac{\alpha}{\sqrt{2}} + j\frac{\Gamma}{\sqrt{2}}\right)\right\}.^{\dagger} \tag{38}$$

Consequently,

$$C_0 = \left(\exp\left(-\frac{\Gamma^2}{2}\right) \operatorname{Re}\left\{\operatorname{erf}\left(\frac{\alpha}{\sqrt{2}} + j\frac{\Gamma}{\sqrt{2}}\right)\right\} + \cos(b) \operatorname{erfc}\left(\frac{\alpha}{\sqrt{2}}\right)\right)^2 \tag{39}$$

can be calculated numerically.[8] However, we can obtain a better analytical understanding of the carrier power by expressing the integral in equation (36) as

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† To make the last step, we used the relations[8]

$$\operatorname{erf}(-z) = -\operatorname{erf}(z) \quad \text{and} \quad \operatorname{erf}(\bar{z}) = \overline{\operatorname{erf}(z)}.$$

$$\exp\left(-\frac{\Gamma^2}{2}\right)\left\{\operatorname{Re}\frac{2}{\sqrt{\pi}}\int_0^{\alpha/\sqrt{2}}\exp\left(-\left(x+j\frac{\Gamma}{\sqrt{2}}\right)^2\right)dx\right\}$$
$$= \exp\left(-\frac{\Gamma^2}{2}\right)\operatorname{Re}\left\{\int_{j(\Gamma/\sqrt{2})}^{\alpha/\sqrt{2}+j(\Gamma/\sqrt{2})}\exp(-z^2)\,dz\right\}. \qquad (40)$$

Since $\exp(-z^2)$ is an entire function, the integral in equation (40) is independent of the path taken from $j(\Gamma/\sqrt{2})$ to $\alpha/\sqrt{2} + j(\Gamma/\sqrt{2})$. It is useful to use a contour consisting of three straight line-segments: the first from $j(\Gamma/\sqrt{2})$ to 0 (on the imaginary axis), the second from 0 to $\alpha/\sqrt{2}$ (on the real axis) and the last from $\alpha/\sqrt{2}$ to $\alpha/\sqrt{2} + j(\Gamma/\sqrt{2})$. Integrating along these lines, it is straightforward to show that

$$\exp\left(-\frac{\Gamma^2}{2}\right)\operatorname{Re}\left\{\frac{2}{\sqrt{\pi}}\int_{j(\Gamma/\sqrt{2})}^{\alpha/\sqrt{2}+j(\Gamma/\sqrt{2})}\exp(-z^2)\,dz\right\}$$
$$= \exp\left(-\frac{\Gamma^2}{2}\right)\operatorname{erf}\left(\frac{\alpha}{\sqrt{2}}\right)$$
$$+ \exp\left(-\frac{\alpha^2}{2}\right)\frac{2}{\sqrt{\pi}}\int_0^{\Gamma/\sqrt{2}}\sin(\sqrt{2}\,\alpha y)\cdot\exp\left(-\left(\frac{\Gamma^2}{2}-y^2\right)\right)dy \qquad (41)$$

Combining equations (33) and (41) yields

$$C_0 = \left(\exp\left(-\frac{\Gamma^2}{2}\right)\operatorname{erf}\left(\frac{\alpha}{\sqrt{2}}\right) + \cos(b)\operatorname{erfc}\left(\frac{\alpha}{\sqrt{2}}\right)\right.$$
$$\left. + \exp\left(-\frac{\alpha^2}{2}\right)\frac{2}{\sqrt{\pi}}\int_0^{\Gamma/\sqrt{2}}\sin(\sqrt{2}\,\alpha y)\exp\left(-\left(\frac{\Gamma^2}{2}-y^2\right)\right)dy\right)^2. \qquad (42)$$

When $\Gamma \ll 1$ (low-index case), the last term inside the brackets is negligible so that

$$C_0 \approx \left(\exp\left(-\frac{\Gamma^2}{2}\right)\operatorname{erf}\left(\frac{\alpha}{\sqrt{2}}\right) + \cos(b)\operatorname{erfc}\left(\frac{\alpha}{\sqrt{2}}\right)\right)^2 \text{ when } \Gamma \ll 1. \qquad (43)$$

It is interesting to observe that $\operatorname{erf}(\alpha/\sqrt{2})$ is the probability that X_t is inside the clipping levels $[-b, b]$ [and $\operatorname{erfc}(\alpha/\sqrt{2})$ is the probability that X_t is outside the clipping levels]. Hence, we can reason that $\exp(-\Gamma^2/2)$ is the average dc voltage associated with $V(t)$ when X_t is inside the clipping levels while $\cos(b)$ is the average voltage when X_t is outside the limits. [One can see from equation (16) that the carrier power $C_0 = |\langle V(t)\rangle|^2$.]

Pursuing these thoughts further, one might conclude that the integral in equation (42) represents a high-order correction (to the preceding approximation) associated with transitions of X_t across the clipping thresholds. When Γ is small, these transitions are rapid and the correction (i.e., the integral) is small. However, as Γ increases, the transitions are slower (at least when X_t is bandlimited) and the integral in equation (42) can increase in magnitude for certain values of α. Moreover,

$$\frac{2}{\sqrt{\pi}} \int_0^{\Gamma/\sqrt{2}} \sin(\sqrt{2}\,\alpha y) \exp\left(-\left(\frac{\Gamma^2}{2} - y^2\right)\right) dy \approx \sin(b) F\left(\frac{\Gamma}{\sqrt{2}}\right) \quad \text{when} \quad \Gamma \gg \sqrt{2},$$

where $F(x) = 2/\sqrt{\pi} \int_0^x \exp[-(x^2 - t^2)]\,dt$ is Dawson's function.[8] Consequently it follows that when $\Gamma \gg \sqrt{2}$ (i.e., the high-index case),

$$C_0 \approx \left(\exp\left(-\frac{\Gamma^2}{2}\right) \operatorname{erf}\left(\frac{\alpha}{\sqrt{2}}\right) + \cos(b) \operatorname{erfc}\left(\frac{\alpha}{\sqrt{2}}\right) + \sin(b) \exp\left(-\frac{\alpha^2}{2}\right) F\left(\frac{\Gamma}{\sqrt{2}}\right)\right)^2. \tag{44}$$

Thus, when α is not too large,

$$C_0 \approx \left(\cos(b) \operatorname{erfc}\left(\frac{\alpha}{\sqrt{2}}\right) + \sin(b) \exp\left(-\frac{\alpha^2}{2}\right) F\left(\frac{\Gamma}{\sqrt{2}}\right)\right)^2. \tag{45}$$

The quantity inside the brackets is a sinusoidal function of α. Hence, there are various values of α (for fixed Γ) which will completely null the carrier power C_0 while other clipping levels give rise to a relative maximum carrier power.

Using

$$\langle V(t) \rangle = \exp\left(-\frac{\Gamma^2}{2}\right) \operatorname{Re}\left\{\operatorname{erf}\left(\frac{\alpha}{\sqrt{2}} + j\frac{\Gamma}{\sqrt{2}}\right)\right\} + \cos(b) \operatorname{erfc}\left(\frac{\alpha}{\sqrt{2}}\right),$$

we computed $\exp(\Gamma^2/2)\langle V(t)\rangle$ as a function of α for several values of Γ. Some of the results are displayed in Fig. 1. (Since the dynamic range of this function is so large, we used a *linear scale* between -1 and 1 on the ordinate axis and a logarithmic scale otherwise.)

The general features of the three functions are as indicated above. Since $\langle V(t)\rangle = 1$ whenever $\alpha = 0$, each curve starts at $\exp(\Gamma^2/2)$ for $\alpha = 0$. For $\Gamma \leqq 1$ (not shown in Fig. 1) the curve decreased monotonically to one. For $\Gamma = 2$, this relative average remains positive but oscillates slightly before converging to unity. However, for the high-index case, the relative mean is similar to a damped sinusoid (as dis-

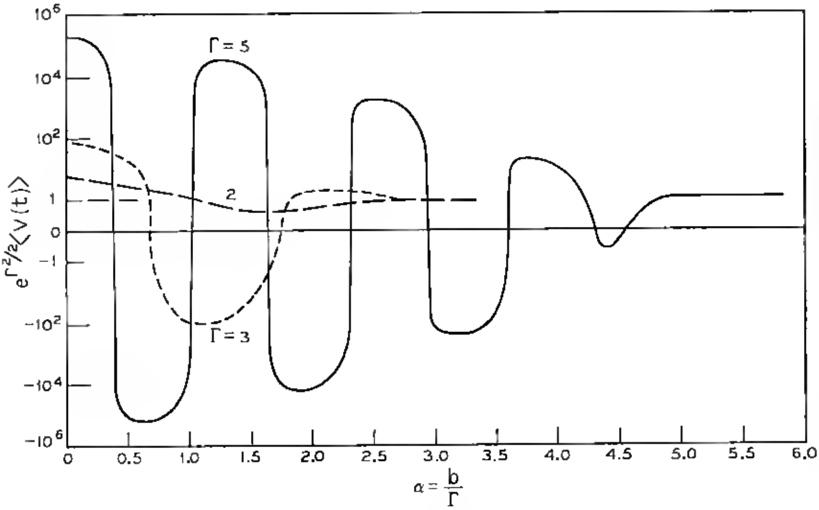

Fig. 1—Relative mean of $V(t)$ as a function of relative clipping level. α is the ratio of the clipping level b to the rms modulation-index Γ.

cussed above). This extreme oscillation about zero apparently occurs for $\Gamma \geqq 3$. Notice that the value of α required for the relative mean (and therefore the carrier power) to stabilize increases as Γ increases. Hence, for high-index modulation, setting $\alpha \approx 4$ does not necessarily guarantee that the carrier power is unaffected by the clipping.

3.2 *Continuous Part of the Spectrum*

Numerical techniques are required to obtain the continuous part of the spectrum. However, one can see from equations (23) and (31) that the coefficients $\{C_n\}$ in the series are independent of $R_x(\tau)$ and need only be computed once for any particular configuration of the hard limiter (and choice of Γ). Consequently, it seems worthwhile to derive an algorithm for efficient computation of $\{C_n\}$.

Noting equations (11), (12) and (13), one can see that

$$R_v(\tau) = \sum_{n=0}^{\infty} \left(\frac{R(\tau)}{\Gamma^2}\right)^n C_n \tag{46}$$

where

$$C_n = \frac{\Gamma^{2n} \mid \nu(n) \mid^2}{n!} \tag{47}$$

and

$$j^n \nu(n) = \int_{-\infty}^{\infty} \exp\left[-jg(x)\right]\left(\frac{d^n}{dx^n} p(x)\right) dx. \tag{48}$$

Consequently,

$$\begin{aligned} j\nu(1) &\\ &= \int_{-b}^{b} \exp(-jx)\, dp(x) + \exp(-jb) \int_{b}^{\infty} dp(x) + \exp(jb) \int_{-\infty}^{-b} dp(x), \\ &= \exp\left(-\frac{\Gamma^2}{2}\right) \operatorname{Re}\left(\operatorname{erfc}\left(\frac{\alpha}{\sqrt{2}} + j\frac{\Gamma}{\sqrt{2}}\right)\right). \end{aligned} \tag{49}$$

Next, for $n \geqq 2$ we have

$$\begin{aligned} j^n \nu(n) = \int_{-b}^{b} \exp(-jx)\, d\left(\frac{d^{n-1}}{dx^{n-1}} p(x)\right) + \exp(-jb) \int_{b}^{\infty} d\left(\frac{d^{n-1}}{dx^{n-1}} p(x)\right) \\ + \exp(jb) \int_{-\infty}^{-b} d\left(\frac{d^{n-1}}{dx^{n-1}} p(x)\right). \end{aligned}$$

Integrating the above expression by parts, we have

$$\nu(n) = \frac{1}{j^{n-1}} \int_{-b}^{b} \exp(-jx)\left(\frac{d^{n-1}}{dx^{n-1}} p(x)\right) dx. \tag{50}$$

Recall that

$$\frac{d^n}{dx^n} \exp\left(-\frac{x^2}{2\Gamma^2}\right) = \left(\frac{-1}{\sqrt{2}\,\Gamma}\right)^n H_n\left(\frac{x}{\sqrt{2}\,\Gamma}\right) \exp\left(-\frac{x^2}{2\Gamma^2}\right)$$

so that integrating equation (50) by parts obtains

$$\nu(n) = \nu(n-1) - \frac{1}{\sqrt{\pi}} \left(\frac{j}{\sqrt{2}\,\Gamma}\right)^{n-1} H_{n-2}\left(\frac{x}{\sqrt{2}\,\Gamma}\right) \exp\left(-jx - \frac{x^2}{2\Gamma^2}\right) \Bigg|_{-b}^{b}. \tag{51}$$

Of course, $H_n(-x) = (-1)^n H_n(x)$, which implies that the exact determination of equation (51) depends on whether n is even or odd. It is straightforward to show that

$$\begin{aligned} \nu(2n) = \nu(2n-1) & \\ + (-1)^n \sin(b) \frac{2}{\sqrt{\pi}} &\exp\left(-\frac{\alpha^2}{2}\right)\left(\frac{1}{\sqrt{2}\,\Gamma}\right)^{2n-1} H_{2(n-1)}\left(\frac{\alpha}{\sqrt{2}}\right) \end{aligned} \tag{52}$$

and

$$\nu(2n+1) = \nu(2n) - (-1)^n \cos(b) \frac{2}{\sqrt{\pi}} \exp\left(-\frac{\alpha^2}{2}\right)\left(\frac{1}{\sqrt{2}\,\Gamma}\right)^{2n} H_{2n-1}\left(\frac{\alpha}{\sqrt{2}}\right)$$

for $n \geqq 1$.

It is known that[8]

$$H_0(x) = 1, \qquad H_1(x) = 2x$$

and

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x) \tag{53}$$

for $n \geqq 1$, so that in principle, the sequence $\{C_n\}$ can be computed by using equations (47), (52) and (53). However, in practice the difference-equations (52) and (53) are unstable and care must be taken in their implementation. We found it best to use the following approach:

Let $[x]$ denote the largest integer not exceeding x and let $n!! = n!/2^{[n/2]}[n/2]!$. [So that $(2n-1)!! = 1 \cdot 3 \cdot 5 \cdots (2n-1)$.] Now, one can use equations (52) and (53) to obtain stable algorithms for

$$t_1(n) = \frac{\Gamma^n \nu(n)}{2^{[n/2]}\left[\dfrac{n}{2}\right]!} \quad \text{and} \quad t_2(n) = \frac{\Gamma^n \nu(n)}{n!!}$$

which are used to compute

$$C_n = t_1(n)t_2(n).$$

3.3 *Numerical Results*

To obtain insight into the effect of hard-limiting, we assume that X_t is bandlimited with a rectangular power spectrum. That is to say

$$R_x(\tau) = \Gamma^2 \frac{\sin 2\pi W\tau}{2\pi W\tau} \quad \text{for} \quad |\tau| < \infty. \tag{54}$$

In this case,[9]

$$S_s(f/W) = C_0\, \delta(f) + \frac{1}{2\pi W} \sum_{n=1}^{\infty} C_n F_n(f/W), \tag{55}$$

for $-\infty < f < \infty$. The functions $F_n(\lambda)$ are defined as follows:[9]

$$F_1(\lambda) = \begin{cases} \pi, & -1 < \lambda < 1, \\ 0, & \text{otherwise}, \end{cases} \tag{56}$$

and for $n \geqq 2$,

$$F_n(\lambda) = \begin{cases} n\pi \displaystyle\sum_{k=0}^{M(n,\lambda)} (-1)^k \frac{\left(\frac{|\lambda| + n}{2} - k\right)^{n-1}}{k!\,(n-k)!}, & 0 \leqq |\lambda| < n, \\ 0, & \text{otherwise,} \end{cases} \tag{57}$$

where $M(n, \lambda) = [n + |\lambda|/2]$. We use $[x]$ to denote the integer part of x.

As $M(n, \lambda)$ increases, it becomes increasingly difficult to accurately sum the alternating series displayed in equation (57). Since

$$F_n(\lambda) \sim \left(\frac{6\pi}{n}\right)^{\frac{1}{2}} \exp\left(-\frac{3\lambda^2}{n}\right) \tag{58}$$

(see Ref. 9), we used the asymptotic approximation (58) whenever $F_n(\lambda)$ was required for $n \geqq 15$. The threshold 15 was determined empirically. We attempted to keep our numerical estimates of the various spectra accurate to one percent relative error.

We considered three cases: low modulation-index ($\Gamma = 0.1$), moderately high index ($\Gamma = 1$) and high index ($\Gamma = 5$). The results are displayed in Figs. 2 through 4. In each case, we have computed the effect of changes in the relative clipping level $\alpha = b/\Gamma$.

3.3.1 *Low Modulation-Index*

When $\Gamma \ll 1$, it is evident from Fig. 2 and equation (55) that the principal part of the spectrum is well approximated by

$$S_v(f/W) \approx \begin{cases} C_0\,\delta(f) + \dfrac{C_1}{2\pi W} F_1(f/W), & \text{for } |f| < W, \\ 0, & \text{otherwise.} \end{cases} \tag{59}$$

Using results from the preceding section, we can show that when $\Gamma \ll 1$, $C_1 \approx \Gamma^2 \exp(-\Gamma^2)(\text{erf}(\alpha/\sqrt{2}))^2$. Hence, for $\Gamma \ll 1$,

$$S_v(f/W) \approx \begin{cases} C_0\,\delta(f) + \left(\text{erf}\left(\dfrac{\alpha}{\sqrt{2}}\right)\right)^2 \dfrac{\Gamma^2 \exp(-\Gamma^2)}{2\pi W} F_1(f/W), & |f| < W, \\ 0, & \text{otherwise.} \end{cases} \tag{60}$$

That is to say, in the low-index case ($\Gamma \ll 1$), hard-limiting causes the principal part of the spectrum to be scaled down by the multiplicative factor $(\text{erf}(\alpha/\sqrt{2}))^2$.

Figure 2 illustrates the actual behavior [as opposed to the approxi-

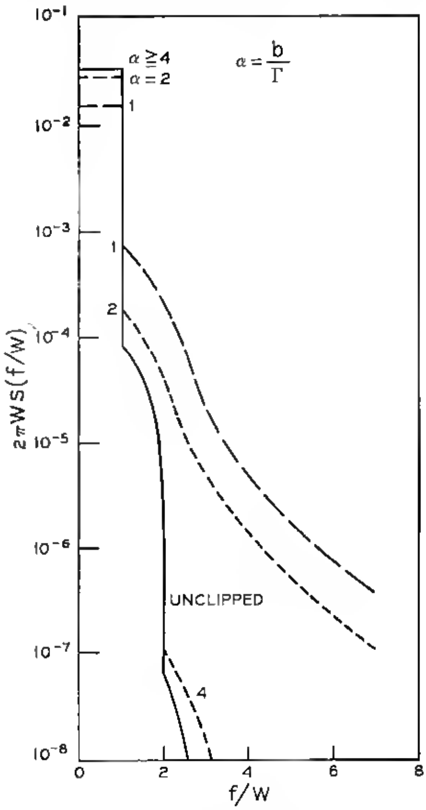

Fig. 2—Spectrum of low-index PM wave ($\Gamma = 0.1$) for various relative clipping levels. α is the ratio of the clipping level b to the rms modulation-index Γ.

mation (60)] of $S_v(f/W)$ for $f > W$. Notice that the tails of the spectrum are raised considerably as the relative clipping level α decreases. This phenomenon is noticeable even for $\alpha = 4$ (i.e., clipping at a four-sigma level). The increase in high-frequency content is apparently caused by the introduction of points at which the derivative of $W(t)$ is discontinuous.

3.3.2 *Moderately High Modulation-Index*

The results for $\Gamma = 1$ are displayed in Fig. 3. The qualitative results are similar to those observed for $\Gamma = 0.1$; i.e., the principal part of the spectrum tends to decrease and the tails increase, as the relative clipping level α decreases. However, notice that for a particular value of α, the frequency f/W at which the tails of the spectrum begin to rise is somewhat larger for $\Gamma = 1$ than for $\Gamma = 0.1$.

3.3.3 *High Modulation-Index*

When $\Gamma = 5$, the behavior of $\langle V(t) \rangle$ as a function of α is quite different from that observed for $\Gamma \leqq 1$ (see Fig. 1). In fact, small changes in α can change the discrete component of the spectrum, $C_0 = \langle V(t) \rangle^2$, from a relative maximum to zero. Consequently, we originally suspected that the continuous part of the spectrum might change significantly as the discrete component changed from a relatively large value to zero. To check this point, four values of α were selected for testing;

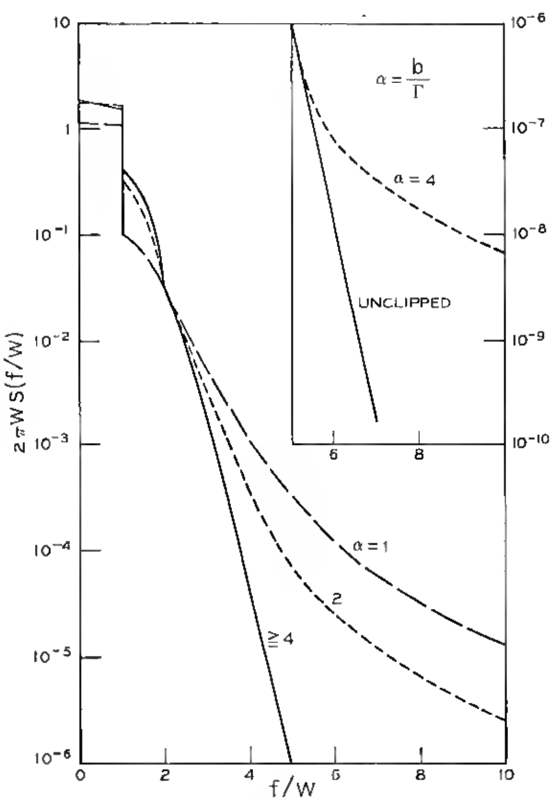

Fig. 3—Spectrum of waves having moderately high modulation-index ($\Gamma = 1$) for various relative clipping-levels. α is the ratio of the clipping level b to the rms modulation-index Γ.

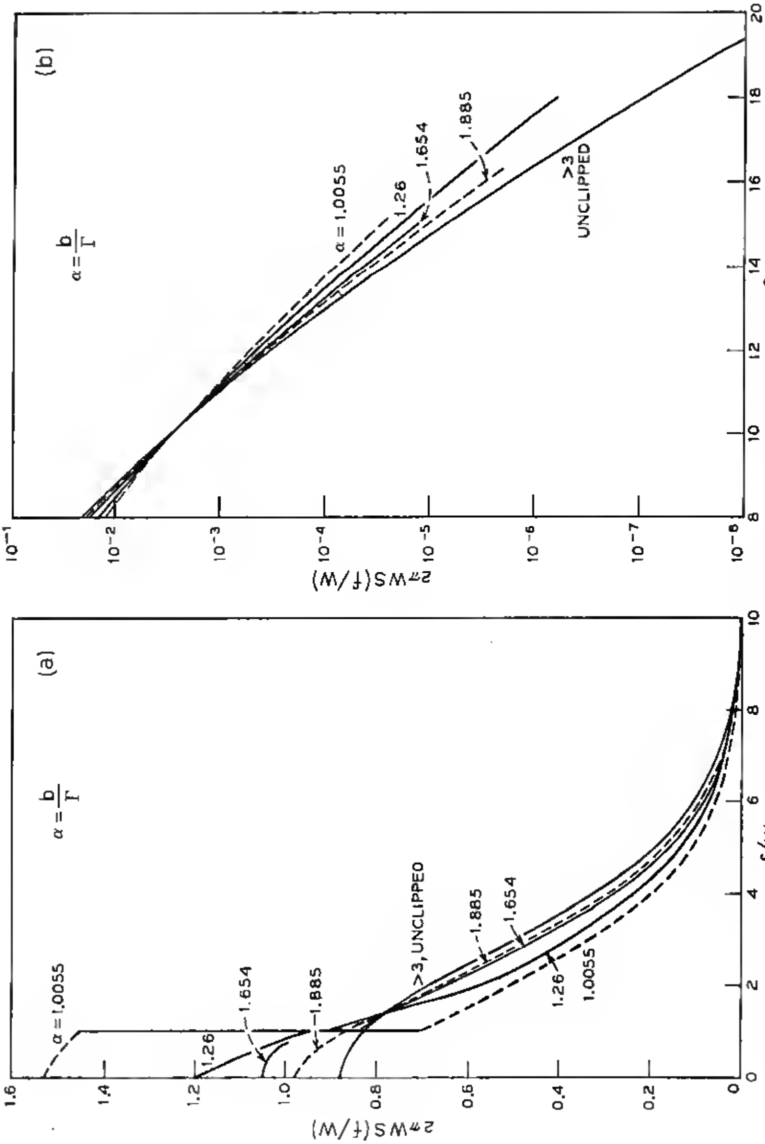

Fig. 4a—Principal part of spectrum for PM waves having high modulation-index ($\Gamma = 5$) for various relative clipping levels. α is the ratio of the clipping level b to the rms modulation-index Γ.

Fig. 4b—Tails of spectrum for PM waves having high modulation-index ($\Gamma = 5$) for various relative clipping levels. α is the ratio of the clipping level b to the rms modulation-index Γ.

two at points of relative maxima for C_0 and two where C_0 was nearly zero. The results are displayed in Figs. 4a and b.

From Figure 4a, we see that the qualitative behavior of $S_v(f/W)$ for high modulation-index ($\Gamma = 5$) differs significantly from that observed for $\Gamma \leqq 1$. However, whether or not C_0 vanishes does not seem to affect the general characteristics of the continuous part of the spectrum. Generally speaking as α decreases, the portion of the spectrum for $0 < f/W < 1$ tends to increase while tbe part for $1 < f/W < 10$ tends to decrease. Notice that a discontinuity is introduced at $f/W = 1$, a characteristic of lower modulation-index. Of course, limiting reduces the variance of the modulating signal X_t, so that the discontinuity at $f/W = 1$ is not surprising.

To examine the behavior of the tails of the spectra, look at Figure 4b. As in the other cases, decreasing α raises the tails of the spectrum, although the effect is considerably less than tbat observed for $\Gamma \leqq 1$. In fact, for $\alpha > 3$, we observed very little difference between the clipped and unclipped cases.

3.4 *Comments*

Computation of the spectrum for large values of f/W, using the infinite series approach, is expensive since many terms are required. Moreover, accurate computation of C_n and F_n, for large n, is difficult (if not impossible) because of numerical problems. Consequently, a good estimate of the spectrum for large f/W would be very desirable. Unfortunately, we do not have such an approximation. However, it is possible to estimate the relative frequency f/W at which the tails of the spectrum for a clipped modulator will begin to depart from the unclipped case.

More precisely, consider equations (48) and (49) which constitute an algorithm for the computation of $\nu(n)$. Since the tails of the spectrum of $W(t)$ can rise only when $\nu(n)$ increases as a function of n, we need to examine $\nu(n)$ for large n.

For large n, we know that[6]

$$H_{2n}(x) = (-1)^n 2^n (2n-1)!! \exp(x^2/2)\left\{\cos(\sqrt{4n+1}\,x) + O\left(\frac{1}{n^{\frac{1}{4}}}\right)\right\} \tag{61}$$

and

$$H_{2n+1}(x) = (-1)^n 2^{n+\frac{1}{2}}(2n-1)!!\sqrt{2n+1} \cdot \exp(x^2/2)\left\{\sin(\sqrt{4n+3}\,x) + O\left(\frac{1}{n^{\frac{1}{4}}}\right)\right\}. \tag{62}$$

Hence, for large n we have the approximation

$$\nu(2n) \approx \nu(2n-1) - 2\Gamma \sin(b)\sqrt{\frac{2}{\pi}} \cos(\alpha\sqrt{2n-3/2}) \cdot \exp\left(-\frac{\alpha^2}{4}\right) \prod_{k=1}^{n-1} \left(\frac{2k-1}{\Gamma^2}\right), \tag{63}$$

$$\nu(2n+1) \approx \nu(2n) + \cos(b)\sqrt{\frac{2}{\pi}} \sin(\alpha\sqrt{2n+3/2}) \cdot \exp\left(-\frac{\alpha^2}{4}\right) \frac{\sqrt{2n+1}}{\Gamma^2} \prod_{k=1}^{n-1} \left(\frac{2k-1}{\Gamma^2}\right). \tag{64}$$

It is evident that the sequence $\mu_n = \prod_{k=1}^{n} [(2k-1))\Gamma]$ is a decreasing function of n for $n < n_0 = [\Gamma^2 + 3/2]$ and is an increasing function for $n > n_0$. For any finite α, μ_n will ultimately exceed $\exp(\alpha^2/4)$ and the forcing function for $\nu(2n)$ will begin to grow without bound. It follows that $\nu(2n)$ will exhibit instability when n is so large that $\mu_n \gg \exp(\alpha^2/4)$. This will certainly be true when $(2n-1)/\Gamma^2 \geqq \exp(\alpha^2/4)$ or for

$$n \geqq \tfrac{1}{2}[\Gamma^2 \exp(\alpha^2/4) + 1]. \tag{65}$$

An inspection of Figs. (2), (3) and (4) shows that equation (65) gives a good indication (at least for those cases considered) of the value of f/W where the tails of the clipped spectra begin a significant departure from the unclipped case. Equations (63) and (64) also indicate that for large n, $\nu(n)$ will oscillate as it increases in magnitude. Consequently, it appears that it will be difficult to get tight bounds on $S_v(f/W)$ for large f/W. It is interesting that the carrier power is most significantly altered by the hard-limiting when $\Gamma \gg 1$ while the effect (of hard-limiting) on the tails of the spectrum decreases as Γ increases.

IV. CONCLUSIONS

In principle, equation (68) can be used to obtain either the autocorrelation function or the spectral density of any "reasonable" function of a stationary gaussian process. We made use of the identity to study the spectral density of a sinusoidal wave which is phase-modulated by a hard-limited gaussian process.

The main disadvantage of this approach (i.e., using an infinite series solution) is that numerical techniques are usually required to obtain a solution. Consequently, estimates of the tails of spectra (of interest for PM systems having high modulation-index) are difficult and expensive to compute accurately. However, when no better alternative is available, our approach can be used to obtain numerical results in a relatively straightforward manner.

V. ACKNOWLEDGMENTS

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APPENDIX

A Relation

Let X_t denote a stationary gaussian process with probability density function

$$p(x) = \frac{1}{2\pi\Gamma^2} \exp\left(-\frac{x^2}{2\Gamma^2}\right), \qquad -\infty < x < \infty. \tag{66}$$

The joint probability density function for $X_{t+\tau}$ and X_t is

$$p(x_1, x_2) = \frac{1}{2\pi\Gamma^2\sqrt{1-r^2}} \exp\left[-\frac{x_1^2 - 2rx_1x_2 + x_2^2}{2\Gamma^2(1-r^2)}\right], \qquad -\infty < x_i < \infty, \tag{67}$$

where

$$R_x(\tau) = \langle X_{t+\tau}X_t\rangle, \qquad \Gamma^2 = R_x(0) \quad \text{and} \quad r = \frac{R_x(\tau)}{\Gamma^2}$$

for $-\infty < \tau < \infty$.

If $G(x)$ is an exponentially bounded complex-valued function of the real variable x (i.e., there are real numbers u and v such that $|G(x)| \leqq v \exp(u\,|x|)$ for $-\infty < x < \infty$) and if $G(x)$ is of bounded variation on finite intervals, then we shall show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x_1)\overline{G(x_2)}p(x_1, x_2)\,dx_1\,dx_2 = \sum_{n=0}^{\infty} \frac{(R_x(\tau))^n}{n!} \left| \int_{-\infty}^{\infty} G(x)\left(\frac{d^n}{dx^n}p(x)\right)dx \right|^2. \tag{68}$$

A.1 *Comments*

All the improper integrals displayed in this Appendix are defined in the sense of principal value. For example, equation (68) is more precisely expressed as

$$\lim_{T_1, T_2 \to \infty} \int_{-T_1}^{T_1} \int_{-T_2}^{T_2} G(x_1)\overline{G(x_2)}p(x_1 , x_2) \, dx_1 \, dx_2$$
$$= \sum_{n=0}^{\infty} \frac{(R_x(\tau))^n}{n!} \left| \lim_{T \to \infty} \int_{-T}^{T} G(x)\left(\frac{d^n}{dx^n} p(x)\right) dx \right|^2 . \tag{69}$$

Since $G(x)$ is exponentially bounded and integrable over $(-T, T)$ for all $T \geqq 0$, it follows that the integrals in equation (69) exist and are bounded. Consequently, the improper integrals in equation (68) also exist. Moreover, one can show that

$$\lim_{T_1 \to \infty} \int_{-T_1}^{T_1} G(x_1)\overline{G(x_2)}p(x_1 , x_2) \, dx_1 = \int_{-\infty}^{\infty} G(x_1)\overline{G(x_2)}p(x_1 , x_2) \, dx_1$$

uniformly for $-\infty < x_2 < \infty$.

It follows that the double limit in equation (69) can be expressed as an iterated limit and further that the improper integral in equation (68) can be evaluated as an iterated improper double integral.

A.2 *Proof of Identity*

We first establish equation (68) for all functions $G(x)$ which are of bounded variation and satisfy

$$\int_{-\infty}^{\infty} | G(x) | \, dx < \infty .$$

In this case (following Rice,[2] Middleton[3] and Bennett[4]), $G(x)$ has a Fourier transform

$$\mathcal{G}(\omega) = \int_{-\infty}^{\infty} G(x) \exp(-j\omega x) \, dx \tag{70}$$

such that for almost all x, $-\infty < x < \infty$,

$$G(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{G}(\omega) \exp(j\omega x) \, d\omega. \tag{71}$$

Using equation (71), we can write

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x_1)\overline{G(x_2)}p(x_1 , x_2) \, dx_1 \, dx_2$$
$$= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \mathcal{G}(\omega_1) \exp(j\omega_1 x_1) \, d\omega_1\right)\left(\int_{-\infty}^{\infty} \overline{\mathcal{G}(\omega_2)} \exp(j\omega_2 x_2) \, d\omega_2\right)$$
$$\cdot p(x_1 , x_2) \, dx_1 \, dx_2 . \tag{72}$$

Since $G(x)$ is of exponential order, one can show that all combinations of the integrals in the right-hand side of equation (72) converge uniformly so that we are justified in interchanging the order of integration to obtain

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x_1)G(x_2)p(x_1, x_2)\,dx_1\,dx_2$$
$$= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{G}(\omega_1)\mathcal{G}(\omega_2)$$
$$\cdot\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x_1, x_2)\exp\left[j(\omega_1 x_1 + \omega_2 x_2)\right]dx_1\,dx_2\right)d\omega_1\,d\omega_2\,. \tag{73}$$

Using the fact that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x_1, x_2)\exp\left[j(\omega_1 x_1 + \omega_2 x_2)\right]dx_1\,dx_2$$
$$= \exp\left\{-\frac{\Gamma^2}{2}(\omega_1^2 + 2r\omega_1\omega_2 + \omega_2^2)\right\}$$

(see Ref. 5, pp. 30–35), and recalling that $\sum_{n=0}^{\infty} x^n/n!$ converges uniformly to $\exp(x)$ for $-\infty < x < \infty$, we obtain the following from equation (73):

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x_1)G(x_2)p(x_1, x_2)\,dx_1\,dx_2$$
$$= \sum_{n=0}^{\infty}\frac{(R_x(\tau))^n}{n!}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}(j\omega_1)^n\mathcal{G}(\omega_1)\exp\left(-\frac{\Gamma^2}{2}\omega_1^2\right)d\omega_1\right\}$$
$$\cdot\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}(j\omega_2)^n\mathcal{G}(\omega_2)\exp\left(-\frac{\Gamma^2}{2}\omega_2^2\right)d\omega_2\right\}. \tag{74}$$

An application of Parseval's theorem yields

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{G}(\omega)(j\omega)^n\exp\left(-\frac{\Gamma^2}{2}\omega^2\right)d\omega = \int_{-\infty}^{\infty} G(-x)\left(\frac{d^n}{dx^n}p(x)\right)dx \tag{75}$$

and a similar expression for $\overline{\mathcal{G}}$ and $\bar{G}$. Using equation (75) in equation (74), we obtain equation (68).

Now, we only assume that $G(x)$ is exponentially bounded and of bounded variation on finite intervals. For $k = 1, 2, \cdots$, define

$$G_k(x) = \begin{cases} G(x), & \text{for} \quad -k \leqq x \leqq k, \\ 0, & \text{otherwise.} \end{cases}$$

It follows that $G_k(x)$ is of bounded variation and that $\int_{-\infty}^{\infty} | G_k(x) | \, dx < \infty$ for $k = 1, 2, \cdots$ so that the above results imply that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_k(x_1)\overline{G_k(x_2)}p(x_1 , x_2) \, dx_1 \, dx_2 = \sum_{n=0}^{\infty} \frac{(R_x(\tau))^n}{n!} \left| \int_{-\infty}^{\infty} G_k(x)\left(\frac{d^n}{dx^n} p(x)\right) dx \right|^2 \tag{76}$$

for $k = 1, 2, \cdots$.

We know that

$$\frac{d^n}{dx^n} p(x) = \frac{1}{\sqrt{2\pi\Gamma^2}} \left(-\frac{1}{\sqrt{2}\ \Gamma}\right)^n H_n\left(\frac{x}{\sqrt{2}\ \Gamma}\right) \exp\left(-\frac{x^2}{2\Gamma^2}\right)$$

[$H_n(z)$ is the Hermite polynomial of degree n], so that the series in equation (76) can be expressed as

$$S_k(r) = \sum_{n=0}^{\infty} a_{n,k}\, r^n \tag{77}$$

where

$$a_{n,k} = \frac{1}{2^n n!} \left| \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} G_k(\sqrt{2}\ \Gamma x) H_n(x) \exp(-x^2) \, dx \right|^2 . \tag{78}$$

Making use of the inequality[8]

$$| H_n(x) | < \xi \exp\left(\frac{x^2}{2}\right) 2^{n/2} \sqrt{n!}, \qquad \xi \approx 1.086435,$$

and the assumption

$$| G(x) | \leqq v \exp(u | x |), \qquad -\infty < x < \infty,$$

we can see that

$$0 \leqq a_{n,k} < \left(\frac{2\xi v}{\sqrt{\pi}} \int_0^{\infty} \exp(u\sqrt{2}\ \Gamma x - x^2) \, dx\right)^2 < \infty,$$

for all n and all k. Hence, when $| r | < 1$, Weierstrass' M-test implies that the series $S_k(r)$ converges uniformly for $k = 1, 2, \cdots$. Consequently, we have

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_k(x_1)\overline{G_k(x_2)}p(x_1 , x_2) \, dx_1 \, dx_2 = \sum_{n=0}^{\infty} \frac{(R_x(\tau))^n}{n!} \left| \lim_{k\to\infty} \int_{-\infty}^{\infty} G_k(x)\left(\frac{d^n}{dx^n} p(x)\right) dx \right|^2 \tag{79}$$

for all τ such tbat $R_x(\tau)/\Gamma^2 \neq \pm 1$; i.e. for $| r | < 1$.

One can also show that

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_k(x_1)\overline{G_k(x_2)}p(x_1\,,\,x_2)\,dx_1\,dx_2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x_1)\overline{G(x_2)}p(x_1\,,\,x_2)\,dx_1\,dx_2\,, \qquad (80)$$

and that

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} G_k(x)\left(\frac{d^n}{dx^n}\,p(x)\right) dx = \int_{-\infty}^{\infty} G(x)\left(\frac{d^n}{dx^n}\,p(x)\right) dx. \qquad (81)$$

Substituting equations (80) and (81) into equation (79) yields the identity (68) for all τ such that $R_z(\tau)/\Gamma^2 \neq \pm 1$.

We saw above that our identity can also be expressed as

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x_1)G(x_2)p(x_1\,,\,x_2)\,dx_1\,dx_2 = \sum_{n=0}^{\infty} a_n r^n, \qquad | r | < 1, \qquad (82)$$

where

$$a_n = \frac{1}{2^n n!} \left| \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} G(\sqrt{2}\,\Gamma x) H_n(x) \exp\,(-x^2)\,dx \right|^2 . \qquad (83)$$

Since the left-hand side of equation (82) exists for $r = \pm 1$ and since $a_n \geqq 0$ for all n, it follows that equation (82) is also valid for $r = \pm 1$. (The proof of this result is a Tauberian theorem. For example, see Ref. 10, page 427, exercise 13–34.)

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